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K-moduli of log del Pezzo pairs and VGIT. (joint with Jesus Martinez-Garcia and Junyan Zhao)

Motivation/History

Aim is to study Fano varieties X or log Fano pairs (X,Δ)
$\mathbb{Q}$-Fano iff $-K_X$ $\mathbb{Q}$-Cartier, $-K_X>0$ $\qquad$ $-K_X-\Delta>0$ iff $-K_X-\Delta$ $\mathbb{Q}$-Cartier

These are fundamental building blocks of varieties (MMP) along with CY and gen. ~~polarized~~ varieties. (Also ties to mirror symmetry)

When X smooth, classification into deformation families in each dim

dim 1 $\quad$ 1 $\quad$ family $\quad$ ($\mathbb{P}^1$)
dim 2 $\quad$ 10 $\quad$ families $\quad$ (del Pezzo surfaces)
dim 3 $\quad$ 105 $\quad$ families $\quad$ (Mori-Mukai, Iskovskikh 80s).
dim 4 > ??
Del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and in general blow-ups of $\mathbb{P}^2$ in $1\le k\le 8$ points. Degree of del Pezzo surface X: $d := (-K_X)^2 = 9-k$

Key goal: construct a moduli theory for Fano varieties/log Fano pairs

Historically for g.p. varieties one has the KSBA moduli space

Need an analogue for Fanos. K-stability has successfully achieved this.

K-stability: AG theory developed to answer when a Fano variety admits a KE metric. Answer: when it is K-polystable. [Chen-Donaldson-Sun 2012]

Due to recent results by various contributors (e.g. Codogni-Blum-Liu-Xu-Wang-Zhuang etc) K-stability constructs a moduli stack $\mathcal{M}^K_{n,V}$ parametrising K-semistable Fano varieties of dimension n and volume $V := (-K_X)^n$.

$\mathcal{M}^K_{n,V}$ admits a good moduli space $M^K_{n,V}$ parametrising K-polystable Fanos of fixed dimension and volume which is a projective, proper scheme.

For log Fano pairs (X, cD) with $D\sim_{\mathbb{Q}} -K_X$ the moduli stack/space $\mathcal{M}^K_{(X,cD)}(c)$, $M^K(c)$ depends on $c\in\mathbb{Q}$.

$\underbrace{\quad}_{(X,D)}$ $\underbrace{\quad}_{(X,D)}$

(Ascher-DeVleming-Liu 19'). $\exists$ finite number of values

$\underline{c_0} < c_1 \leq \dots < \underline{c_k}$ such that

$\underbrace{\mathcal{M}^{K}_{(X,D)}(c)}\cong \underbrace{\mathcal{M}^{K}_{(X,D)}(c')}$ for all $c, c' \in \underbrace{(c_i, c_j)}$, but

$\mathcal{M}_{(X,D)}(c_i) \not\cong \mathcal{M}_{(X,D)}(c_j)$ for all $i \neq j$.

The interval (c_i, c_j) is called a <u>chamber</u>
the values c_i are called <u>walls</u>.
This is modeled after GIT where the same phenomenon is observed depending on polarization (Thaddeus, Dolgachev-Hu 90's)

Previous known examples: $\underbrace{(\mathbb{P}^2, cC)}$ (Ascher-DeVleming-Liu 19'), $(\underbrace{\mathbb{P}^1\times\mathbb{P}^1}, cC)$ (ADL 20')
$(\underbrace{\Sigma_5}, cD)$ where $\underbrace{D \sim -2K_X}$ $\underbrace{\text{(Zhao 23')}}$. etc...

<u>Today</u>: Describe $\underbrace{\mathcal{M}^K_{\underline{d}}(c)}$ for del Pezzo surfaces of degree $\underline{d \geq 2}$ and $\underbrace{D \sim -K_X}$.
(X, cD).

<u>Main theorem</u> (Martinez-Garcia - P. - Zhao. 24')
1) Let $d = 3, 4$. Then $a = 3$, $b = 4$

$$\underbrace{\mathcal{M}^K_d(c)} \cong \underbrace{\mathcal{M}^{GIT}_d(t(c))} \text{ and } \underbrace{M^K_d(c) \cong M^{GIT}_d(t(c))},\ \in .$$

where: a) $\mathcal{M}^{GIT}_3(t(c))$ is the VGIT moduli stack of

GIT semistable pairs (S, H) s.t. $S \subset \mathbb{P}^3$ is cubic and $H \subset \mathbb{P}^3$ is a hyperplane s.t.
$D := H \cap S \sim -K_S$. $\underbrace{t(c) = \frac{9c}{8+c}}$ a

b) $\underbrace{\mathcal{M}^{GIT}_4(t(c))}$ is the VGIT moduli stack of

GIT semistable pairs (S, H) s.t. $\underline{S = Q_1 \cap Q_2} \subset \mathbb{P}^4$ is a complete intersection of two quadrics and $\underline{H \subset \mathbb{P}^4}$ is a hyperplane s.t.
$D := H \cap S \sim -K_S$. $t(c) = \frac{6c}{5+c}$.

In both cases, there are six walls.

2) Let $d = 2$. Let $M^{GIT}_{\mathbb{P}^2}(t)$, be the VGIT quotient parametrizing pairs $[(C_4, L)]$ where $C_4 \subset \mathbb{P}^2$ is a quartic curve in $\mathbb{P}^2$ and $L \subset \mathbb{P}^2$ is a line.

$\emptyset$

Then $\underbrace{M^K_2(c)} \cong \mathrm{Bl}_{\underbrace{[(2C_2, L)]}} M^{GIT}_{\mathbb{P}^2}(t(c))$ where $t = 2c$.

3) $d \geq 4$ No VGIT picture, explicit description

<u>Degrees 3, 4</u> (for brevity I will only describe $d = 4$. $d = 3$ is very similar)

smooth del Pezzo surface of degree 4: blow-up of $\mathbb{P}^2$ at 5 points (in general position). Alternatively:

complete intersection of two quadrics Q_1, Q_2 in $\mathbb{P}^4$.

Mabuchi-Mukai '94 Odaka-Spotti-Sun 15 described K-moduli space of dP_4 by relating it to GIT: they showed

$\overline{M}^K_4 \cong \overbrace{Gr(2,15)}/\!\!/\underbrace{PGL(5)}$. Here $\mathrm{Pic}\, Gr \cong \mathbb{Z}$ so only one choice of polarisation.

Used mainly analytical techniques (as a moduli space of KE varieties).

Let us consider log Fano pairs (Σ_4, cD) where $D \sim -K_{\Sigma_4}$. $0 < c < 1$.

By adjunction $D = \Sigma_4 \cap H$, is a hyperplane section.

Natural Question: Is the K-moduli stack/space $\mathcal{M}^K_4(c) / M^K_4(c)$ also described by a GIT quotient?

[P. 22'] considered VGIT quotient

$\mathcal{M}^{GIT}_4(t) = \underbrace{\overset{Q_1 \cap Q_2}{Gr(2,15)}} \times \overset{H}{\overbrace{\mathbb{P}^4}} /\!\!/_t\, PGL(5)$. Here $\underline{\mathrm{Pic}\,(Gr \times \mathbb{P}) = \mathbb{Z}^2}$

choice of linearisation not unique $\Rightarrow$ VGIT problem.

$t = \frac{b}{a}$

Described explicitly the GIT quotient in each wall/chamber using a computational algorithm.

<u>Theorem</u> [P. 22'] If a pair (Σ_4, cD) is K-(semi/poly) stable then the pair (Σ_4, H) is GIT (semi/poly) stable. $D = \Sigma_4 \cap H$.

Furthermore, $\underbrace{t = \frac{6c}{5+c}}$ and

$\underbrace{\mathcal{M}^K_4(c)} \cong \underbrace{\mathcal{M}^{GIT}_4} (t(c))$ for all $\underline{c < c_2}$.

New key Lemma (X_0, cD_0)

<u>Lemma</u> (MG-P. - 2) Let (X_0, cD_0) be a K-semistable log del Pezzo pair admitting a $\mathbb{Q}$-Gorenstein smoothing to $\underline{(\Sigma_4, cD)}$, and $D \in |-K_{\Sigma_4}|$. Then X_0 can be embedded to $\mathbb{P}^4$ as a complete intersection of two quadrics.

To prove the main theorem: we use the results from [P. 22'] and the above Lemma in an inductive step to show that the required isomorphism extends to all walls/chambers.

For $d=3$ we have similar result, (Gallardo-Martinez-Garcia - Spotti 19')

Remark The results of [P.22'] and [C-MG-S] are <u>necessary</u> to run the inductive step. If we didn't have them it would be impossible to prove the main theorem.

degree 2 case

Smooth del Pezzo surface of degree 2: Blow-up of $\mathbb{P}^2$ at 7 points (in general position). Alternatively:

$\pi: \Sigma_2 \longrightarrow \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ branched along a quartic curve C_4.

2C.

GSS: showed that $\underbrace{M_2^K}_{} \simeq \underbrace{Bl_{[2C]}}_{} \overbrace{\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)))}^{\text{quartics } (14, S.)} // PGL(3)$.

Here 2C is the double conic. This is a Kirwan-blow up of the GIT quotient along a polystable orbit. Locally looks like

$\{(\underbrace{\mathbb{P}^{14} // PGL(3)}_{}) \setminus \{\underbrace{[2C]}_{}\}\} \cup E$ where E is the exceptional divisor of the blow up where E parametrizes GIT polystable octic curves in $\mathbb{P}(1,1,4)$ under a PGL_2-action.

In our setting: Instead of considering pairs (X, cD) we consider pairs

$\boxed{(\mathbb{P}^2, \frac{1}{2}C_4 + cL)}$. Why? By RH $\pi: X \to \mathbb{P}^2$

: $\underbrace{-K_X = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))}_{}$ where $\pi: X \to \mathbb{P}^2$ is the double cover

Step 1:
Lemma 1 Let $(X, \frac{1}{2}C + cC')$ be a K-semistable pair which admits (by def being) Q-Gorenstein smoothing to $\underline{(\mathbb{P}^2, \frac{1}{2}C_4 + cL)}$. Then either

$X \simeq \mathbb{P}^2$ $(C = \{f_4 = 0\}, \ C' = \{f_1 = 0\})$

or

$X \simeq \mathbb{P}(1,1,4)$ $(C = \{f_8 = 0\}, \ C' = \{f_2 = 0\})$ $\quad M_2^K(c) \quad X \nearrow \mathbb{P}^2, \ \searrow \mathbb{P}(1,1,4)$

Step 2:
Theorem 1 A pair $(X, \frac{1}{2}C + cC')$ is K-polystable if and only if $\underline{(Y, cD)}$ is K-polystable where $\pi: Y \to X$ double cover branched along C, and $D = \pi^* C'$.

Theorem 1 justifies our choice of pairs. Lemma 1 tells us that the K-moduli stack contains pairs with either $\mathbb{P}^2$ or $\mathbb{P}(1,1,4)$.

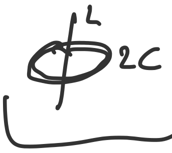
Step 3:
The next step is to describe the VGIT quotient

We consider the VGIT quotient $(C_4, L) \subset \mathbb{P}^2$

$H_4 \times H_1 // PGL := \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(4))) \times \overbrace{\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(1)))}^{\text{lines in } \mathbb{P}^2} // PGL$

Using the computational methods in [P.22'], [Gallardo-Martinez-Garcia-Zanardini 18]

we explicitly classify this quotient based on singularities (also partially described by Laza 06). In particular:

The pair 

is GIT polystable for all walls/chambers

<u>Step 4</u>: Relate VGIT with K-stability.

<u>Theorem 2</u>: Let (C_4, L) be a GIT (semi/poly)stable pair, where $C_4 \neq 2C$. Then the pair $(\mathbb{P}^2, \frac{1}{2}C_4 + cL)$ is K-semi/polystable for $t = 2c$.

This theorem is proven by computing the invariants defining K-stability.

<u>Step 5</u>: Perform the Kirwan blow-up $Bl_{[2C,L]} H_4 \times H_1 /\!/_t PGL_3$.

We show that $[(2C, L)]$ allows us to construct a Luna slice.

In particular we obtain a universal family

$$f : (\mathcal{Z}, \tfrac{1}{2}\mathcal{C} + c\,\mathcal{C}_1) \longrightarrow Bl_{[2C,L]} \underbrace{H_4 \times H_1} =: \underbrace{U_t} \quad (\text{exc. divisor } \mathcal{E})$$

<u>Proposition 3</u> The family f is

i) isomorphic to $\underbrace{(\mathbb{P}^2, \tfrac{1}{2}\mathcal{C}_4 + c\,\mathcal{C}_1)|_{H_4 \times H_1 \setminus [(2C,L)]}} \longrightarrow \underbrace{H_4 \times H_1 \setminus [(2C,L)]}$

over the open locus $\underbrace{U_t \setminus \mathcal{E}}$

ii) the fibres over $\mathcal{E}$ are of the form $(\mathbb{P}(1,1,4), \tfrac{1}{2}C_8 + C_2)$, with coordinates x, y, z.

<u>Step 6</u>: We consider the VGIT quotient

$$\mathcal{M}^{GIT}_{\mathbb{P}^1} = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))) \times \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))) /\!/ PGL_2.$$

(Note: in $\mathbb{P}(1,1,4)$, $f_8 = z^2 + \underbrace{2f_4(x,y)} + f_8(x,y)$ so we can perform a change of coordinates to write $f_8 = z^2 + \underbrace{f_8(x,y)}$, similarly $f_2 = f_2(x,y)$ so the above parametrizes the pairs we need.

<u>Theorem 4</u> A pair $(\mathbb{P}(1,1,4), \tfrac{1}{2}C_8 + cC_2)$ is K-(semi/poly)stable if and only if the pair (C_8, C_2) is GIT-$t(c)$ (semi/poly)stable with $\underbrace{t(c) = \frac{12c}{1-c}}$; $C_8 = \{z^2 + f_8 = 0\}$, $C_2 = \{f_2 = 0\}$.

$f_8 = \tilde{C}_8$, $f_2 = \tilde{C}_2$, $\tilde{C}_2 = C_2$.

<u>Final step</u> combining the above, we obtain the desired isomorphism

$Bl_{[2C,L]} \mathcal{H}_4 \times \mathcal{H}_1 /\!/ PGL_3$ "=" $(\mathcal{H}_4 \times \mathcal{H}_1 /\!/ PGL_3 \setminus \{[2C,L]\}) \cup \mathcal{E}$

$\mathcal{E} \cong \mathcal{M}^{GIT}_{\mathbb{P}^1}$

<u>Higher degrees</u> $d > 4$ Here there is no explicit VGIT like no we find walls explicitly via numerical criteria.

$d = 9$ no walls (ADL 19) $(\mathbb{P}^2, cC)$

$d = 8$ and $\Sigma_8 = \mathbb{P}^1 \times \mathbb{P}^1$ one wall $\underbrace{c = 1/4}$. <u>ADL 23</u> $D \sim \mathcal{O}(2,2)$.

$d = 8$ and $\underbrace{\Sigma_8 = Bl_p \mathbb{P}^2}$ two walls $\underbrace{c_1 = 1/5}$, $\underbrace{c_2 = 1/4}$ (Pan-Si-Wu 23')

$d = 5$ 6 walls (Zhao 24')

these two were described for $\underbrace{D \sim -2K_X}$. Since a pair

$\underbrace{(X, cD)}$ is $K(s/p)$ stable iff $\boxed{(X, \underbrace{\tfrac{c}{2}\, 2D})}$ is $K(s/p)$ stable

we obtain the description very easily.

we explicitly find the walls for $d = 6, 7$.

<u>Theorem</u> (JC-P-Z 24').

1) If <u>$d = 7$</u> there are three walls and <u>X can have up to A_1 singularities</u>.

2) If <u>$d = 6$ there are five walls</u> and <u>X can have up to A_2 singularities</u>.

Method of proof:

we first determine that $d = 7$ X is smooth or has 1 A_1 sing.

$d = 6$ X is smooth or has A_1 or A_2 sings.

then compute invariants for all possibilities of D.

<u>Remark</u> $d = 1$ similar to $d = 2$ case much harder, takes more time ongoing!